# ON SUBGROUPS OF FINITE SOLUBLE GROUPS III

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ABSTRACT

We define a notion of generalized normality for subgroups of finite soluble groups. We show that a normalizer relative to this notion exists and is homomorphism-invariant. We make comparisons with previous constructions, and develop briefly a general theory of normality relations and normalizers.

In this paper we consider only finite soluble groups; the word *group* and the letter $G$ are reserved to denote such a group, while $H$ always denotes a subgroup of $G$.

In the two preceding papers with the same title [5, 6], we introduced, for any $G$ and $H$, some subgroups of $G$ which are *generalized normalizers* of $H$ in $G$ in some sense. In this paper we describe yet one more such construction. The subgroup $X_G(H)$ which we introduce is the *normalizer* of $H$ defined by a relation which may be described as *the most general abstract normality relation.* The subgroup $X_G(H)$ shares with $Q_G(H)$ (in [5]) the property of being homomorphism invariant. However, the construction of $X_G(H)$ is independent of previous results and seems to us to be simpler than the construction of $Q_G(H)$.

We also discuss briefly *abstract normality relations* in the second half of the paper.

Notation and terminology are mostly standard. We use $H^G$ and $H_G$, respectively, to denote the normal closure and normal core of $H$, that is, the smallest normal subgroup containing $H$ and the largest normal subgroup contained in $H$. $H$ sn $G$ means that $H$ is a subnormal subgroup of $G$.

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DEFINITION. Let $H$ be a subgroup of the (finite soluble) group $G$. $H$ is $X$-normal in $G(H \text{ xn } G)$ if the following condition holds:
(1) Given an epimorphism of $G$, if $H^\sigma \neq G^\sigma$, then $H^\sigma$ is contained in a proper normal subgroup of $G^\sigma$.

In other words, $H$ xn $G$ if, for all $N \Delta G$, $HN \neq G$ implies $H^G N = (HN)^G \neq G$.

PROPOSITION 1. (i). *If $H$ xn $G$ and $\sigma$ is an epimorphism of $G$, then $H^\sigma$ xn $G^\sigma$.*

(ii) *If $\sigma$ is an epimorphism of $G$ such that $H$ contains the kernel of $\sigma$, and $H^\sigma$ xn $G^\sigma$, then $H$ xn $G$.*

(iii) *If $H$ xn $K$ xn $G$, then $H$ xn $G$.*

(iv) *$H$ is not $X$-normal in $G$ if and only if there exists an $N \Delta G$, such that $HN$ is an abnormal maximal subgroup of $G$.*

(v) *$H$ is $X$-normal in $G$ if and only if, for every abnormal maximal subgroup $M \supseteq H$, we have $M \neq HM_G$.*

PROOF. (i), (ii) and (iii) being trivial, and (v) being a reformulation of (iv), we prove only the latter.

First, if for some $N \Delta G$, $HN$ is abnormal and maximal, then $HN \neq G$ and $(HN)^G = G$, so $H$ is not $X$-normal in $G$. Conversely, let $H$ be not $X$-normal, and choose $N$ maximal such that $N \Delta G$, $HN \neq G$ but $(HN)^G = G$. Let $M/N$ be a chief factor of $G$. If $HM \neq G$, then by maximality of $N$, we have $(HN)^G \subseteq (HM)^G \neq G$, a contradiction. Thus $HM = G$, which means that $HN$ complements the chief factor $M/N$, and therefore $HN$ is a maximal subgroup of $G$, which is abnormal, since $(HN)^G = G$.

DEFINITION. A subgroup $K \supseteq H$ is an *X-normalizer* of $H$ in $G$ (denoted $K = X_G(H) = X(H)$), if

(i) $H$ xn $K$, and

(ii) if $H$ xn $L \subseteq G$, then $L \subseteq K$.

THEOREM 2. *Every subgroup of $G$ possesses an X-normalizer in $G$.*

FIRST PROOF. By induction on $|G|$. Let $N$ be a minimal normal subgroup of $G$. Let $L/N = X_{G/N}(HN/N)$, which exists by induction. If $L \neq G$, then, by induction, $K = X_L(H)$ exists, and it is easy to check that $K = X_G(H)$. Thus we assume that for all minimal normal subgroups $N$ of $G$, $HN$ xn $G$.

If, for some such $N$, $HN = G$, then $H$ is maximal in $G$ (we assume $H \neq G$), so either $H \Delta G$ and $G = X_G(H)$ or $H$ is abnormal in $G$, in which case $H = X_G(H)$.

Thus we may assume also that $HN \neq G$, for each minimal normal $N$. Taking one such $N$, we have $HN$ xn $G$, $HN \neq G$, thus $H^G \subseteq (HN)^G \neq G$. Next, if $\{1\} \neq M \Delta G$ and $HM \neq G$, we may pick a minimal normal $N \subseteq M$, hence $HN$ xn $G$ implies $HM$ xn $G$ and $(HM)^G \neq G$. Thus $H$ xn $G$ and $G = X_G(H)$.

SECOND PROOF. We may assume that $H$ is not $X$-normal in $G$. Pick an abnormal maximal subgroup $M$ such that $M = HM_G$ (Proposition 1(v)): if $H$ xn $L$, then $M$ xn $LM_G$ implies $M = LM_G$, that is, $L \subseteq M$. Now we can apply induction to $M$ or proceed as follows. Denote, $X_1(H; G) = \bigcap M$ ($M$ maximal and abnormal in $G$ and $M = HM_G$), $X_{i+1}(H; G) = X_1(H; X_i(G))$.

Then it follows that if $H$ xn $L$, then $L \subseteq X_i(H; G)$, for all $i$. There exists some $n$ for which $X_n(H; G) = X_{n+1}(H; G)$, and then one verifies that $X_n(H; G) = X_G(H)$.

Thus we also have a procedure for constructing $X_G(H)$. A variant on this construction is obtained by deleting *maximal* in the definition of $X_1$.

COROLLARY 3. *Any Sylow system $S$ of $G$ that reduces into $H$ also reduces into $X_G(H)$.*

PROOF. Choose $M$ as in the second proof. (If $X_G(H) = G$ there is nothing to prove.) Then $M = HM_G$ implies that $S$ reduces into $M$[1, Cor. 2.8]. It follows that $S$ reduces into $X_1(H; G)$ [9, Prop. 9], and now we repeat the argument to derive that $S$ reduces into $X_2(H; G)$, and so forth.

THEOREM 4. *The $X$-normalizer is epimorphism invariant, that is, if $\sigma$ is an epimorphism of $G$, then $X_{G^\sigma}(H^\sigma) = (X_G(H))^\sigma$.*

PROOF. By induction. Let $G^\sigma = G/N$, and $X_{G/N}(HN/N) = L/N$; then $X_G(H) \subseteq L$. If $L \neq G$, we apply induction to $L$. Therefore we assume $HN$ xn $G$, but $H$ is not $X$-normal in $G$. Find an $R \Delta G$ such that $HR \neq G$ and $(HR)^G = G$. Now $HN$ xn $G$ implies $HRN$ xn $G$, hence if $HRN \neq G$ we obtain

$$(HR)^G \subseteq (HRN)^G \neq G.$$

Thus $HRN = G$. Denote $M = HR$; then $M \neq G$. But $M^\sigma = G^\sigma$, hence by induction

$$G^\sigma = X_{G^\sigma}(H^\sigma) = X_{M^\sigma}(H^\sigma) = (X_M(H))^\sigma \subseteq (X_G(H))^\sigma,$$

$$G^\sigma = (X_G(H))^\sigma. \qquad \text{Q.E.D.}$$

COROLLARY 5. *$X_G(H)$ is abnormal in $G$.*

PROOF. Assume $X_G(H) \subseteq L \Delta M \subseteq G$. Then in $M/L$ one has

$$HL/L = \{1\} \text{ xn } M/L,\ X_M(H)L/L = \{1\}$$

contradicting Theorem 4. Thus, whenever, $M \supseteq L \supseteq X_G(H)$, $L$ is not normal in $M$, which implies that $X_G(H)$ is abnormal [3, VI. 11.17].

As an illustration, we consider some subgroups of groups of small nilpotent length.

THEOREM 6. (i) *Let $G = EF$, where $E$ and $F$ are nilpotent subgroups of $G$, and $F \Delta G$. Then $X_G(E)$ is a Carter subgroup of $G$.*

(ii) *Let $D$ be a system normalizer of $G$, where $G$ has nilpotent length 3 at most; then $X_G(D)$ is a Carter subgroup of $G$.*

PROOF. (i) It is known that $E \subseteq C$, where $C$ is a Carter subgroup of $G$ [8, Lemma 1]. Let $N$ be a minimal normal subgroup of $G$. By induction, $X_G(E)N = CN$. If $CN \neq G$, we conclude by applying induction in $CN$. Thus assume $CN = G$, hence $G/N$ is nilpotent, for each minimal normal subgroup $N$ of $G$. If there are at least two minimal normal subgroups, we find that $G$ itself is nilpotent, and everything is trivial. Thus we assume that $N$ is the unique minimal normal sugroup of $G$, and that $C \neq G$; hence $N = F(G)$, and therefore $F \subseteq N$ and $G = EN = CN$. However, $C \cap N = \{1\}$ (or else $C \supseteq N$ and $G = C$ is nilpotent), hence $E = C$ and $E = X_G(E)$.

(ii). Let $F = F(G)$. Then $DF = CF$ [3, VI. 12.4], and we use (i) in $CF$.

Now let us indicate some possible generalizations and applications of our results.

I. Let us suppose that we are given some notion of abstract normality, that is, we are given a relation $H$ an $G$, which holds between some of the subgroups of $G$ and $G$ itself. We postulate the following conditions.

(2) If $H$ an $G$, and $\sigma$ is an epimorphism of $G$, then $H^\sigma$ an $G^\sigma$.

(3) If $H$ is a maximal subgroup of $G$, then $H$ an $G$ if and only if $H \Delta G$.

Examples of such relations are: normality, subnormality, $X$-normality, and the relation $Q(H) = G$, where $Q(H)$ is the reducer of $H$, discussed in [5, 6].

PROPOSITION 7. *For any abstract normality relation, if* $H$ an $G$, *then* $H$ xn $G$.

Thus $X$-normality is the most general abstract normality relation.

PROOF. If $H$ is not $X$-normal in $G$, we choose a subgroup $M$ as in Proposition 1(iv). Then $HM_G/M_G = M/M_G$ is not abstract normal in $G/M_G$, hence $H$ is not abstract normal in $G$.

A further natural condition to impose is that

(4) $H$ an $G$ and $H \subseteq K \subseteq G$ implies $H$ an $K$.

An easy induction shows that, for a normality relation satisfying (2), (3) and (4), $H$ an $G$ implies $H$ sn $G$ so that (4) is a very strong condition and we refrain from imposing it.

Usually, a normalizer (defined in the obvious way) relative to a given abstract normality does not exist; for example, a subnormalizer usually does not exist. One can always pass from the given relation to one for which a normalizer exists, such as the relation: $G$ is generated by subgroups $L$ such that $H$ an $L$.

Suppose, next, that our relation satisfies (2), (3) and the following statement.

(5) An abstract normalizer $A_G(H)$ exists, and is epimorphism invariant.

An argument similar to that of Corollary 5 shows that in this case $H$ sn $G$ implies $H$ an $G$. It follows, that the functor $A_G(H)$ satisfies the following

(6) $A_G(H) \supseteq N_G(H)$.

(7) If $\sigma$ is an epimorphism of $G$, then $A_{G^\sigma}(H^\sigma) = (A_G(H))^\sigma$.

(8) If $H \subseteq L \subseteq G$, then $A_L(H) \subseteq A_G(H)$.

By [6, Th. 2], we now have $A_G(H) \supseteq Q_G(H)$. Every functor satisfying (6)–(8), and the following statement,

(9) if $H$ is an abnormal maximal subgroup of $G$ then $H = A_G(H)$,

is the abstract normalizer functor defined by the relation $G = A_G(H)$. Thus $Q$ and $X$ are the extreme points of the set of functors satisfying (6)–(9).

II. Let $H$ be a Schunck class, that is, a non-empty class of groups satisfying the following statements:

(10) If $G \in H$ and $\sigma$ is an epimorphism of $G$, then $G^\sigma \in H$.

(11) If $G/M_G \in H$ for all maximal subgroups $M$ of $G$, then $G \in H$.

A maximal subgroup $M$ of a group $G$ is termed an *$H$-normal subgroup*, if $G/M_G \in H$. We define *$HX$-normality* by changing, in condition $X$, the phrase "proper normal subgroup" to "$H$-normal maximal subgroup". Then existence and epimorphism-invariance of $HX$-normalizers are established as before. We can also give a discussion similar to I above, only in (3) we change "$H \Delta G$" to $H$ is "$H$-normal in $G$", and so forth. (A similar generalization of the functor $Q_G(H)$ was given by Graddon [2].)

III. The subgroup $X_G(H)$ can replace the subgroup $Q_G(H)$ in several constructions. We have in mind the $Q$-series of [6], the construction of Carter subgroups from system normalizers [4, Sec. 4], and the proof in [7].

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